A simple monodimensional model coupling an enthalpy transport equation and a neutron diffusion equation

Stéphane Dellacherie, Erell Jamelot, Olivier Lafitte

A simple monodimensional model coupling an enthalpy transport equation and a neutron diffusion equation

Stéphane Dellacherie[a,b,c], Erell Jamelot[a], Olivier Lafitte[a,d]

[a] *Commissariat à l'Énergie Atomique et aux Énergies Alternatives, CEA, DEN, DM2S, F-91191 Gif-sur-Yvette, France.*
[b] *Université Pierre et Marie Curie (Paris 6), LRC-Manon, Laboratoire J.L. Lions, 4 place Jussieu, 75005 Paris, France.*
[c] *École Polytechnique de Montréal, C.P. 6079, succ. Centre-ville, Montréal (Québec), H3C 3A7, Canada.*
[d] *Université Paris 13, Sorbonne Paris Cité, LAGA, CNRS (UMR 7539), 99 Avenue J.-B. Clément F-93430, Villetaneuse Cedex, France.*

Abstract

We obtain an analytic solution of a monodimensional stationary system coupling a simplified thermohydraulic model to a simplified neutronic model based on the diffusion approximation with one energy group. We obtain this solution with minimal hypotheses on the absorption and fission cross sections, and on the diffusion coefficient.

Key words:
Ordinary differential equation, operator spectrum, thermohydraulics, neutronics, models coupling.

1. Introduction

In this Note, we construct an analytic solution of the low Mach number thermohydraulic model

$$\begin{cases} \frac{d}{dz}(\rho u) = 0, & \text{(a)} \\ \frac{d}{dz}(\rho u^2 + \pi) = \rho g, & \text{(b)} \\ \rho u \dfrac{d}{dz} h = \mathbb{E}\Sigma_f(h)\phi(t,z) & \text{(c)} \end{cases} \tag{1}$$

coupled to the simplified neutronic model based on the diffusion approximation with one energy group

$$-\frac{d}{dz}\left[D(h)\frac{d}{dz}\phi(z)\right] + \left[\Sigma_a(h) - \frac{\nu\Sigma_f(h)}{k_{eff}}\right]\phi(z) = 0. \tag{2}$$

In (1) and (2), $z \in [0, L]$ is the spatial variable, $L > 0$ being the length of the nuclear core. Moreover, in (1), $\rho(z)$, $u(z)$, $\pi(z)$ and $h(z)$ are respectively the density, the velocity, the dynamical pressure and the internal enthalpy of the flow. The source terme ρg is a volumic force (*e.g.* the gravity field). The constant $\mathbb{E}$ is the energy released by a fission ($\mathbb{E} > 0$ is in Joule), $\Sigma_f(h)$ is the fission (macroscopic) cross section ($\Sigma_f(h) > 0$ is in m^{-1}) and $\phi(z)$ – solution of (2) – is the scalar neutron flux ($\phi(z) \geq 0$ is in $\mathrm{m}^{-2}\cdot\mathrm{s}^{-1}$). In (2), $D(h)$ is the diffusion coefficient ($D(h) > 0$ is in m), $\Sigma_a(h)$ is the absorption (macroscopic) cross section ($\Sigma_a(h) > 0$ is in m^{-1}) and ν is the average number of neutron produced by a fission. Moreover, the density ρ and the internal enthalpy h are linked through the equation of state $\rho = \varrho(h)$ where $\varrho(\cdot)$ is a given function[1]. At last, $k_{eff} > 0$ is the neutron multiplication factor: $k_{eff} \in]0, 1[$, $k_{eff} = 1$ and $k_{eff} > 1$ means that the nuclear core is respectively subcritical, critical and supercritical.

Email addresses: `stephane.dellacherie@cea.fr` (Stéphane Dellacherie), `erell.jamelot@cea.fr` (Erell Jamelot), `lafitte@math.univ-paris13.fr` (Olivier Lafitte)

[1] The fact that the equation of state $\varrho(h)$ depends only on h is a consequence of the low Mach regim [1].

Using (1), we obtain $\rho u = D_e$ where $D_e > 0$ is a positive constant defining the flow rate. Thus, (1)(c) and (2) give the simplified thermohydraulics-neutronics system

$$\begin{cases} D_e \dfrac{dh}{dz} = \mathbb{E}\Sigma_f(h)\phi, & \text{(a)} \\ -\dfrac{d}{dz}\left[D(h)\dfrac{d\phi}{dz}\right] + \left[\Sigma_a(h) - \dfrac{\nu\Sigma_f(h)}{k_{eff}}\right]\phi = 0. & \text{(b)} \end{cases} \tag{3}$$

We supplement this system, written for $\phi \in H_0^1([0, L])$ (which means that ϕ satisfies homogeneous Dirichlet boundary conditions $\phi(0) = \phi(L) = 0$)[2] and $h \in C^1([0, L])$, with the constraint $\phi \geq 0$ on $[0, L]$ and with the boundary conditions

$$h(0) = h_e \quad \text{and} \quad h(L) = h_s. \tag{4}$$

Let us note that knowing $h(z)$, the density $\rho(z)$ is given by $\rho(z) = \varrho[h(z)]$. This allows to obtain the velocity $u(z)$ with $u(z) = \frac{D_e}{\rho(z)}$. At last, the dynamical pressure $\pi(z)$ is obtained by integrating (1)(b) and by using the boundary condition $\pi(L) = \pi_*$ where π_* is the pressure at the outlet of the nuclear core. In [3], we construct an analytical solution of (3)(4) when $D(h)$ and $\Sigma_f(h)$ are positive constants, $\Sigma_a(h)$ being a non-constant function of h (to enforce the coupling). In this Note, we generalize this result by supposing that $D(h)$ and $\nu\Sigma_f(h)$ are also functions depending on h.

The outline of this Note is the following. In Section 2, we construct an analytical solution of (3)(4). In Section 3, we underline the link between (3)(4) and an eigenvalue problem. Then, we conclude the Note.

2. Construction of an analytical solution

To construct an analytical solution of (3)(4), we assume that the given functions $\Sigma_f(h)$, $\Sigma_a(h)$ and $D(h)$ verify the two following hypotheses:

Hypothesis 1. *The enthalpy always belongs to a fixed domain $[h_{min}, h_{max}]$ on which $\Sigma_f(h)$, $\Sigma_a(h)$ and $D(h)$ are continuous functions.*

Hypothesis 2. *There exists $\alpha_f > 0$, $\alpha_a > 0$ and $\alpha_d > 0$ such that in $[h_{min}, h_{max}]$*

$$\Sigma_f(h) \geq \alpha_f, \quad \Sigma_a(h) \geq \alpha_a \quad \textit{and} \quad D(h) \geq \alpha_d. \tag{5}$$

Hypothesis 2 is somewhat too strong for physical applications. For example, there exists zones in a nuclear core where there is no fission and, thus, where Σ_f is equal to 0.

Using a classical property of Sturm-Liouville operators [2], we have the following result:

Lemma 2.1. *Under Hypotheses 1 and 2, for any $(h_e, h_s) \in [h_{min}, h_{max}]^2$, there exists a unique function $\mathcal{X} \in C^1([h_e, h_s])$ solution of*

$$\begin{cases} -\dfrac{d}{dh}\left[D(h)\nu\Sigma_f(h)\dfrac{d\mathcal{X}}{dh}\right] = \dfrac{\Sigma_a(h)}{\nu\Sigma_f(h)}, \\ \mathcal{X}(h_e) = \mathcal{X}(h_s) = 0 \end{cases} \tag{6}$$

and there exists a unique function $\mathcal{Y} \in C^1([h_e, h_s])$ solution of

$$\begin{cases} -\dfrac{d}{dh}\left[D(h)\nu\Sigma_f(h)\dfrac{d\mathcal{Y}}{dh}\right] = 1, \\ \mathcal{Y}(h_e) = \mathcal{Y}(h_s) = 0. \end{cases} \tag{7}$$

Moreover, $\mathcal{X}$ and $\mathcal{Y}$ are positive on $]h_e, h_s[$.

[2] This is the natural set-up for this second order elliptic equation, which is thus written in the weak sense.

Let us note that we cannot write that $(\mathcal{X},\mathcal{Y}) \in \left(C^2([h_e,h_s])\right)^2$ since Σ_f and D are *a priori* only continuous on $[h_e,h_s]$. Nevertheless, with an *ad hoc* change of variable, one has extra regularity. Indeed, under Hypothesis 1 and 2, we can define the C^1-diffeomorphism

$$\theta(h) = \int_{h_e}^{h} \frac{dh'}{D(h')\nu\Sigma_f(h')} \tag{8}$$

from $[h_e,h_s]$ to $[0,\theta_{\max}]$ with $\theta_{\max} = \theta(h_s)$. Then, Equations (6) and (7) become $-\frac{d^2x}{d\theta^2} = f(\theta)$ and $-\frac{d^2y}{d\theta^2} = g(\theta)$ with $x(\theta) = \mathcal{X}(h(\theta))$, $y(\theta) = \mathcal{Y}(h(\theta))$, $f(\theta) = D(h(\theta))\Sigma_a(h(\theta))$, $g(\theta) = D(h(\theta))\nu\Sigma_f(h(\theta))$ and with the boundary conditions $x(0) = x(\theta_{\max}) = y(0) = y(\theta_{\max}) = 0$. This new functions satisty the following regularity results:

Lemma 2.2. *Under Hypotheses 1 and 2, x and y belong to $C^2([0,\theta_{max}])$. Moreover, $\frac{x}{y}$ belongs to $C^1([0,\theta_{max}]) \cap C^2(]0,\theta_{max}[)$. At last, $y(\theta) = (\theta_{max}-\theta)\theta r(\theta)$ with $r(\theta) \geq \frac{1}{2}\nu\alpha_f\alpha_d > 0$ and $r \in C^0([0,\theta_{max}])$.*

Let us note that due to Hypothesis 2, we obtain that x and y are concave functions on $[0,\theta_{\max}]$, positive on $]0,L[$ and such that $x'(0) > 0$, $y'(0) > 0$, $x'(\theta_{\max}) < 0$ and $y'(\theta_{\max}) < 0$.

Proof of Lemma 2.2: The first item comes from the continuity of the functions f and g under Hypothesis 1. Moreover

$$x(\theta) = -(\theta_{max}-\theta)\int_0^1 x'[\theta_{max}+t(\theta-\theta_{max})]dt, y(\theta) = -(\theta_{max}-\theta)\int_0^1 y'[\theta_{max}+t(\theta-\theta_{max})]dt, \tag{9}$$

$$y(\theta) = (\theta_{max}-\theta)\theta\int_0^1\int_0^1 tg[t\theta_{max}+tu(\theta-\theta_{max})]dtdu. \tag{10}$$

Since $y'(\theta_{\max}) < 0$, one deduces from (9) that $\frac{x}{y} = \frac{\int_0^1 x'[\theta_{max}+t(\theta-\theta_{max})]dt}{\int_0^1 y'[\theta_{max}+t(\theta-\theta_{max})]dt}$ belongs to $C^1\left([\frac{\theta_{max}}{2},\theta_{max}]\right)$. A similar equality shows that $\frac{x}{y}$ belongs to $C^1\left([0,\frac{\theta_{max}}{2}]\right)$. We deduce from (10) that $r(\theta) = \int_0^1\int_0^1 tg[t\theta_{max}+tu(\theta-\theta_{max})]dtdu$. Thus, $r \in C^0([0,\theta_{max}])$ and, under Hypothesis 2, we find that r is bounded below by $\frac{1}{2}\nu\alpha_f\alpha_d$.□

Lemma 2.2 allows us to prove the main result:

Theorem 2.1. *Let us define $k_\infty := \min_{h\in[h_e,h_s]} \frac{\mathcal{Y}(h)}{\mathcal{X}(h)} = \frac{\mathcal{Y}(h_*)}{\mathcal{X}(h_*)}$. Under Hypotheses 1 and 2, there exists a unique solution $(h,\phi,k_{eff}) \in C^1([0,L]) \times H_0^1([0,L]) \times]0,k_\infty[$ of (3)(4) such that $\phi(z) \geq 0$ on $[0,L]$. Moreover, $\phi(z) > 0$ on $]0,L[$ and $\phi \in C^1([0,L])$.*

The regularity of h (resp. ϕ) is *a priori* not better that $C^1([0,L])$ because we only suppose that $\Sigma_f(h)$ (resp. $D(h)$) is continuous. Before proving Theorem 2.1, let us note that the function $\phi \geq 0$ is unique in the chosen formulation (3)(4). Nevertheless, thanks to the enthalpy equation (3)(a) which imposes the relation $D_e(h_s-h_e) = \mathbb{E}\int_0^L \Sigma_f(h(z))\phi(z)dz$, System (3)(4) can be rewritten

$$\left\{\begin{aligned} &\frac{dh}{dz} = (h_s-h_e)\frac{\mathbb{E}\Sigma_f(h)\phi}{\displaystyle\int_0^L \mathbb{E}\Sigma_f(h(z))\phi(z)dz},\\ &-\frac{d}{dz}\left[D(h)\frac{d\phi}{dz}\right] + \left[\Sigma_a(h) - \frac{\nu\Sigma_f(h)}{k_{eff}}\right]\phi = 0 \end{aligned}\right. \tag{11}$$

again coupled to the boundary conditions (4). This new system – which is often used in the field of thermohydraulics-neutronics coupling – is invariant with respect to the transformation $\phi \mapsto \mu\phi$ where μ is a positive constant.

The proof of Theorem 2.1 is a consequence of Lemma 2.2 and also of the two following results:

Proposition 2.1. *Let us suppose that (3)(4) admits a solution (h,ϕ,k_{eff}) belonging to $C^1([0,L]) \times H_0^1([0,L]) \times \mathbb{R}_+$ and such that $\phi(z) \geq 0$ on $[0,L]$. Then:*

1. $k_{eff} \in]0, k_\infty]$. *Moreover, k_{eff} is solution of*

$$L = \mathcal{L}(k_{eff}) := \int_{h_e}^{h_s} \frac{dh}{\nu\Sigma_f(h)\sqrt{2\left[\frac{1}{k_{eff}}\mathcal{Y}(h) - \mathcal{X}(h)\right]}}. \tag{12}$$

2. *Once k_{eff} is obtained, $h(z)$ is solution of*

$$\int_{h_e}^{h(z)} \frac{dh'}{\nu\Sigma_f(h')\sqrt{2\left[\frac{1}{k_{eff}}\mathcal{Y}(h') - \mathcal{X}(h')\right]}} = z. \tag{13}$$

3. *Once $h(z)$ is obtained, $\phi(z)$ is given by*

$$\phi(z) = \frac{\nu D_e}{\mathbb{E}}\sqrt{2\left[\frac{1}{k_{eff}}\mathcal{Y}(h) - \mathcal{X}(h)\right]}. \tag{14}$$

Let us underline that simple algebra allows to obtain that k_{eff} solution of (12) is such that $k_{eff} \to k_{eff,0} := \frac{\nu\Sigma_f(h_e)}{\frac{\pi^2 D(h_e)}{L^2} + \Sigma_a(h_e)}$ when $h_s \to h_e$. Thus, we recover the classical neutron multiplication factor $k_{eff,0}$ obtained when there is no coupling with the thermohydraulics [4].

Lemma 2.3. *Under Hypotheses 1 and 2, the function $\mathcal{L}(k)$ is defined on $[0, k_\infty[$ and is an increasing one to one function that maps $[0, k_\infty[$ into $[0, +\infty[$.*

By using this lemma, we obtain the existence and unicity of (h, ϕ, k_{eff}) constructed in Proposition 2.1 (with $k_{eff} \in]0, k_\infty[$ since $L > 0$ and $L < +\infty$). Using Hypothesis 1, relations (13) and (14) prove that $(h, \phi) \in C^1([0, L]) \times H_0^1([0, L])$. And since $k_{eff} \in]0, k_\infty[$, we have $\frac{1}{k_{eff}}\mathcal{Y}(h) - \mathcal{X}(h) > 0$ on $]h_e, h_s[$ that is to say $\phi(z) > 0$ on $]0, L[$. Moreover, (3)(b) implies that $D(h)\phi' \in C^1([0, L])$. Thus, $\phi' \in C^0([0, L])$ that is to say $\phi \in C^1([0, L])$. This ends the proof of Theorem 2.1.

Proof of Proposition 2.1: Since $\phi(z) = \frac{D_e \frac{dh}{dz}}{\mathbb{E}\Sigma_f(h)}$, (3)(b) is given by

$$-\frac{d}{dz}\left[D(h)\frac{d\phi}{dz}\right] + \frac{\nu D_e}{\mathbb{E}}\left[\frac{\Sigma_a(h)}{\nu\Sigma_f(h)} - \frac{1}{k_{eff}}\right]\frac{dh}{dz} = 0.$$

Thus, using (6) and (7), we find

$$-\frac{d}{dz}\left[D(h)\frac{d\phi}{dz}\right] - \frac{\nu D_e}{\mathbb{E}}\left\{\frac{d}{dh}\left[D(h)\nu\Sigma_f(h)\frac{d\mathcal{X}}{dh}\right] - \frac{1}{k_{eff}}\frac{d}{dh}\left[D(h)\nu\Sigma_f(h)\frac{d\mathcal{Y}}{dh}\right]\right\}\frac{dh}{dz} = 0.$$

Using the chain rule, we recognize the (weak)[3] derivative with respect to z of $D(h)\nu\Sigma_f(h)\frac{d\mathcal{X}}{dh} - \frac{1}{k}D(h)\nu\Sigma_f(h)\frac{d\mathcal{Y}}{dh}$. Hence, as the only distributions u satisfying $u' = 0$ are the constants, there exists a constant K_0 such that

$$-D(h)\frac{d\phi}{dz} - \frac{\nu D_e}{\mathbb{E}}\left[D(h)\nu\Sigma_f(h)\frac{d\mathcal{X}}{dh} - \frac{1}{k_{eff}}D(h)\nu\Sigma_f(h)\frac{d\mathcal{Y}}{dh} - K_0\right] = 0.$$

We multiply this equation by the continuous function $\frac{\phi(z)}{D(h)} = \frac{D_e \frac{dh}{dz}}{\mathbb{E}D(h)\Sigma_f(h)}$ which gives

$$-\phi(z)\frac{d\phi}{dz} - \left(\frac{\nu D_e}{\mathbb{E}}\right)^2\left[\frac{d\mathcal{X}}{dh} - \frac{1}{k_{eff}}\frac{d\mathcal{Y}}{dh} - \frac{K_0}{D(h)\nu\Sigma_f(h)}\right]\frac{dh}{dz} = 0.$$

[3] We have only here a weak derivative because, although $\mathcal{X}$ is of class $C^1([h_e, h_s])$, $D(h)\nu\Sigma_f(h)$ is *a priori* only continuous.

Again, using the chain rule and the (weak) derivative of ϕ^2, defining $G(h) := \int_{h_e}^{h} \frac{1}{D(s)\nu\Sigma_f(s)} ds$, one obtains that there exists a constant K_1 such that $-\frac{1}{2}\phi^2(z) - \left(\frac{\nu D_e}{\mathbb{E}}\right)^2 \left[\mathcal{X}(h) - \frac{1}{k_{eff}}\mathcal{Y}(h) - K_0 G(h) - K_1\right] = 0$. The boundary conditions $\phi(0) = \phi(L) = 0$ and $\mathcal{X}(h_e) = \mathcal{X}(h_s) = \mathcal{Y}(h_e) = \mathcal{Y}(h_s) = 0$ yield $K_0 = K_1 = 0$. Finally, by replacing $\phi(z)$ with $\frac{D_e \frac{dh}{dz}}{\mathbb{E}\Sigma_f(h)}$, we obtain $\frac{1}{2}(h'(z))^2 = (\nu\Sigma_f(h))^2 \left[\frac{1}{k_{eff}}\mathcal{Y}(h) - \mathcal{X}(h)\right]$. A necessary condition on the solution h is thus $\frac{1}{k_{eff}}\mathcal{Y}(h(z)) - \mathcal{X}(h(z)) \geq 0$ on $[0, L]$ that is to say $k_{eff} \in]0, k_\infty]$ ($k_{eff} \neq 0$ since $L \neq 0$). Let us now suppose that $\phi(z) \geq 0$. Thus, $h'(z) \geq 0$. When $k_{eff} \in]0, k_\infty[$, this allows to deduce from the previous relation that

$$\frac{h'(z)}{\nu\Sigma_f(h)\sqrt{2\left[\frac{1}{k_{eff}}\mathcal{Y}(h) - \mathcal{X}(h)\right]}} = 1. \tag{15}$$

Hence (13) since $h(0) = h_e$. Relation (12) is a consequence of (13) by also using $h(L) = h_s$. At last, $\phi(z) = \frac{D_e \frac{dh}{dz}}{\mathbb{E}\Sigma_f(h)}$ and (15) leads to (14).□

Proof of Lemma 2.3: The proof comes from the two following observations:
i) $\mathcal{L}(k) = \sqrt{k}\int_{h_e}^{h_s} \frac{dh}{\nu\Sigma_f(h)\sqrt{2[\mathcal{Y}(h) - k\mathcal{X}(h)]}} = \sqrt{k}\int_0^{\theta_{\max}} \frac{D(h(\theta))d\theta}{\sqrt{2[y(\theta) - kx(\theta)]}} \leq \alpha_d\sqrt{\frac{k}{2}}\int_0^{\theta_{\max}} \frac{d\theta}{\sqrt{y(\theta) - kx(\theta)}}$ where $\theta(h)$ is given by (8) and $\theta_{\max} = \theta(h_s)$. On the other side, Lemma 2.2 implies that the behaviour of $\int_0^{\theta_{\max}} \frac{d\theta}{\sqrt{y(\theta)}}$ is given by the behaviour of $\int_0^{\theta_{\max}} \frac{d\theta}{\sqrt{\theta(\theta_{max}-\theta)}}$, which proves that $\int_0^{\theta_{\max}} \frac{d\theta}{\sqrt{y(\theta)}} < +\infty$. As a consequence, $\mathcal{L}(k) \to 0$ when $k \to 0$ which, by continuity, allows to write that $\mathcal{L}(0) = 0$.

ii) $\frac{x}{y}$ is continuous on $[0, \theta_{max}]$ (see Lemma 2.2), hence admits a maximum denoted by $\frac{1}{k_\infty}$. Two cases are possible:
ii.1) There exists a value $\theta(h_*) \in]0, \theta_{max}[$ such that $\frac{x}{y}[\theta(h_*)] = \frac{1}{k_\infty}$. In this case, there exists $\delta > 0$ and $m_0 > 0$ such that $[\theta(h_*) - \delta, \theta(h_*) + \delta] \subset]0, \theta_{\max}[$ and

$$\forall\theta \in [\theta(h_*) - \delta, \theta(h_*) + \delta] : \quad \frac{1}{k} - \frac{1}{k_\infty} \leq \frac{1}{k} - \frac{x(\theta)}{y(\theta)} \leq \frac{1}{k} - \frac{1}{k_\infty} + \frac{1}{2}m_0[\theta - \theta(h_*)]^2. \tag{16}$$

Inequation (16) is a consequence of $\frac{x}{y} \in C^2(]0, \theta_{max}[)$ and of the existence of $\tilde{\theta}[\theta, \theta(h_*)] \in [\theta(h_*) - \delta, \theta(h_*) + \delta]$ such that $\frac{x(\theta)}{y(\theta)} = \frac{1}{k_\infty} + \frac{1}{2}[\theta - \theta(h_*)]^2 \left(\frac{x}{y}\right)''(\tilde{\theta})$ thanks to the Taylor-Lagrange formula. Finally, since we have also $\mathcal{L}(k) = \int_0^{\theta_{\max}} \frac{D(h(\theta))d\theta}{\sqrt{2y(\theta)\left[\frac{1}{k} - \frac{x(\theta)}{y(\theta)}\right]}}$, we obtain (using a lower bound on $\frac{1}{\sqrt{y(\theta)}}$) that there exists $m > 0$ such that $\mathcal{L}(k) \geq m\int_{\theta(h_*)-\delta}^{\theta(h_*)+\delta} \frac{d\theta}{\sqrt{\frac{1}{k} - \frac{1}{k_\infty} + \frac{1}{2}m_0[\theta - \theta(h_*)]^2}} =: J(k, \delta)$. On the other side, by using the change of variable $\theta = \theta(h_*) + \sqrt{\frac{2}{m_0}\left(\frac{1}{k} - \frac{1}{k_\infty}\right)}\, t$, we find that $J(k, \delta) = 2m\sqrt{\frac{2}{m_0}}\int_0^{\beta(k,\delta)} \frac{dt}{\sqrt{1+t^2}}$ with $\beta(k, \delta) := \frac{\delta}{\sqrt{\frac{2}{m_0}\left(\frac{1}{k} - \frac{1}{k_\infty}\right)}} \to +\infty$ when $k \to k_\infty$. Hence, $J(k, \delta) \to +\infty$ when $k \to k_\infty$ which implies that $\mathcal{L}(k) \to +\infty$ when $k \to k_\infty$.
ii.2) The maximum is obtained at θ_{max}. In this case, one writes $\frac{x}{y}(\theta) = \frac{1}{k_\infty} + (\theta - \theta_{max})\left(\frac{x}{y}\right)'[\tilde{\theta}(\theta, \theta_{max})]$, with $\left(\frac{x}{y}\right)'[\tilde{\theta}(\theta, \theta_{max})] \geq 0$ on $[\theta_{max} - \delta, \theta_{max}]$. Hence, there exists m_0 such that $\frac{1}{k} - \frac{x}{y}(\theta) \geq \frac{1}{k} - \frac{1}{k_\infty} + m_0(\theta_{max} - \theta)$ that is to say $y(\theta)\left[\frac{1}{k} - \frac{x}{y}(\theta)\right] \geq (\theta_{max} - \theta)\theta r(\theta)\left[\frac{1}{k} - \frac{1}{k_\infty} + m_0(\theta_{max} - \theta)\right]$ (r is defined in Lemma 2.2). Thus, using a lower bound of $m_0\theta r(\theta)$ on $[\theta_{max} - \delta, \theta_{max}]$, we find $M > 0$ such that $\mathcal{L}(k) \geq M\int_{\theta_{max}-\delta}^{\theta_{max}} \frac{d\theta}{\sqrt{(\theta_{max}-\theta)\left[\frac{1}{m_0}\left(\frac{1}{k} - \frac{1}{k_\infty}\right) + \theta_{max} - \theta\right]}} = M'\int_0^{\frac{\delta m_0}{\frac{1}{k} - \frac{1}{k_\infty}}} \frac{dt}{\sqrt{t + t^2}}$ (where M' is another positive constant). Hence, $\mathcal{L}(k) \to +\infty$ when $k \to k_\infty$. The case where the maximum is reached at $\theta = 0$ is treated similarly. □

3. Eigenvalue problem associated with the solution of the coupled system

We now increase the regularity of $\Sigma_f(h)$ by supposing that:

Hypothesis 3. *The function $\Sigma_f(h)$ belongs to $C^1([h_{min}, h_{max}])$.*

This allows to obtain the following result:

Proposition 3.1. *Assume Hypotheses 1–3 and let (h, k_{eff}) be the solution obtained in Theorem 2.1. Then:*

i) The operator

$$P(h) := -(\nu\Sigma_f(h))^{-\frac{1}{2}}\frac{d}{dz}\left\{D(h)\frac{d}{dz}\left[(\nu\Sigma_f(h))^{-\frac{1}{2}}\right]\right\} + \frac{\Sigma_a(h)}{\nu\Sigma_f(h)}$$

is a self-adjoint positive operator belonging to $\mathcal{L}[H_0^1([0,L]), H^{-1}([0,L])]$.

ii) The real $\frac{1}{k_{eff}}$ is the smallest eigenvalue of $P(h)$.

iii) The function $(\nu\Sigma_f(h))^{\frac{1}{2}}\phi$ where ϕ is given by (14) is an eigenvector of $P(h)$ associated to the eigenvalue $\frac{1}{k_{eff}}$.

Proof of Proposition 3.1: We have

$$\langle P(h(z))\phi_1, \phi_2\rangle = \int_0^L D(h(z))\frac{d}{dz}\left[(\nu\Sigma_f(h(z)))^{-\frac{1}{2}}\phi_1\right]\frac{d}{dz}\left[(\nu\Sigma_f(h(z)))^{-\frac{1}{2}}\phi_2\right]dz + \int_0^L \frac{\Sigma_a(h)}{\nu\Sigma_f(h)}\phi_1\phi_2 dz = \langle \phi_1, P(h(z))\phi_2\rangle$$

which proves that $P(h)$ is self-adjoint. Moreover, we have $\frac{d}{dz}\left[(\nu\Sigma_f(h(z)))^{-\frac{1}{2}}\psi\right] = (\nu\Sigma_f(h(z)))^{-\frac{1}{2}}\left[\frac{d\psi}{dz} - \psi(z)h'(z)\frac{(\nu\Sigma_f(h(z)))'}{2\nu\Sigma_f(h(z))}\right]$. Since $h \in C^1([0,L])$, using Hypothese 2 and the Poincaré inequality, we can find a positive constant C such that $\langle P(h(z))\psi, \psi\rangle \geq C\int_0^L \left(\frac{d\psi}{dz}\right)^2 dz$ for any $\psi \in H^1([0,L])$. Hence, we find that the operator $P(h)$ is a coercive self-adjoint operator in $H^1([0,L])$. Through the Lax-Milgram Theorem, $P(h)$ extends to a bounded bicontinuous operator from $H_0^1([0,L])$ to $H^{-1}([0,L])$. The classical theory of Sturm-Liouville operators [2] ensures that there is an increasing sequence of eigenvalues $\lambda_1, ..., \lambda_n, ..$, associated with normalized eigenvectors $\psi_1, ..., \psi_n, ...$. The unique eigenvector of constant sign is ψ_1. Let us now come back to the unique solution (h, ϕ, k_{eff}) given by Theorem 2.1 and let us define $\psi(z) = (\nu\Sigma_f(h(z)))^{\frac{1}{2}}\phi(z)$. The problem rewrites $-\frac{d}{dz}\left\{D(h)\frac{d}{dz}\left[(\nu\Sigma_f(h))^{-\frac{1}{2}}\right]\right\}\psi = \frac{1}{k_{eff}}(\nu\Sigma_f(h(z)))^{\frac{1}{2}}\psi(z)$ that is to say $P(h)\psi = \frac{1}{k_{eff}}\psi$. Moreover, since $\nu\Sigma_f(h(z)) \in C^1([0,L])$ and $\phi \in H_0^1([0,L])$, we have $\psi \in H_0^1([0,L])$. This proves that ψ is an eigenvector for $P(h)$ associated to the eigenvalue $\frac{1}{k_{eff}}$. We conclude by noting that ψ is positive (since by construction, $\phi \geq 0$) and thus belongs to the unique eigenspace $\mathbb{R}\psi_1$ composed of constant sign eigenfunctions. As a consequence, $\frac{1}{k_{eff}}$ is the smallest eigenvalue of $P(h)$.$\square$

4. Conclusion

We have constructed in this Note an analytic solution of a simplified stationary thermohydraulics-neutronics model with minimal hypotheses on the absorption and fission cross sections, and on the diffusion coefficient. The construction of this solution underlines in particular that the thermohydraulics-neutronics coupling is not an eigenvalue problem although it is possible to prove that, when the internal enthalpy is known, the scalar neutron flux is also solution of an eigenvalue problem. Nevertheless, since this coupling is non-linear, the internal enthalpy cannot be known independently of the scalar neutron flux. Even though one recovers the classical set-up of neutronic equations (where the flux is an eigenvector and $\frac{1}{k_{eff}}$ is the smallest eigenvalue), it is only a *a posteriori* result for the coupling problem.

References

[1] S. Dellacherie – *On a low Mach nuclear core model* – ESAIM:PROC, 35, pp. 79-106, 2012. 1

[2] E.A. Coddington and N. Levinson – *Theory of ordinary differential equations* – Chapter 8, International series in pure and applied mathematics, McGraw-Hill, 1955. 2, 6

[3] S. Dellacherie and O. Lafitte – *Une solution explicite monodimensionnelle d'un modèle simplifié de couplage stationnaire thermohydraulique-neutronique* – Preprint HAL-01263642v1, 2016. Available at https://hal.archives-ouvertes.fr/hal-01263642v1. 2

[4] J.J. Duderstadt and L.J. Hamilton – *Nuclear Reactor Analysis* – Wiley & Sons, New York, 1976. 4